Invariant Measures in Time-Delay Coordinates for Unique Dynamical System Identification

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Acknowledgement: Robert Martin[3]

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MS146: Scientific Machine Learning for Stable Prediction of Dynamical Systems

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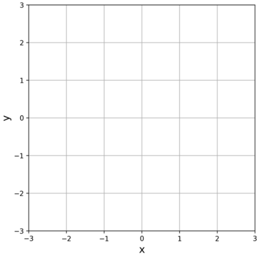
Dynamical System Identification

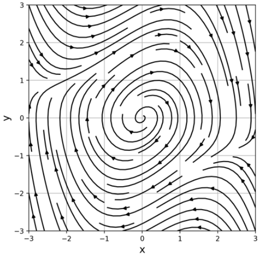
$$X$$

State space



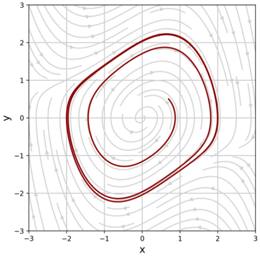
$$\dot{x} = v(x)$$

Evolution rule



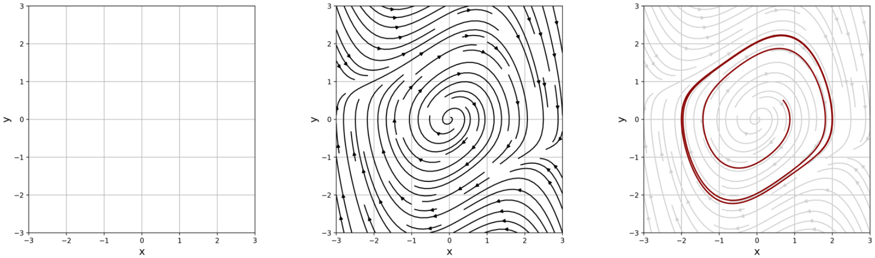
$$\{x(t_k)\}_{k=0}^{N-1}$$

Trajectory samples



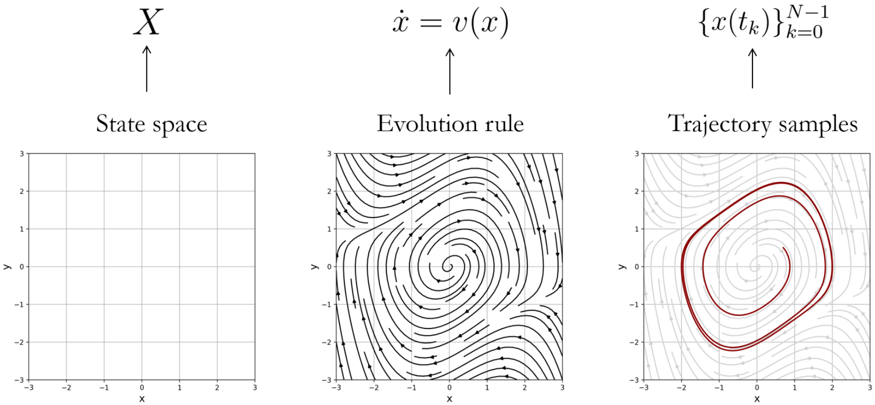
Dynamical System Identification

X	$\dot{x} = v(x)$	$\{x(t_k)\}_{k=0}^{N-1}$
$\uparrow$	$\uparrow$	$\uparrow$
State space	Evolution rule	Trajectory samples



- We parameterize $v(x;\theta)$ by $\theta \in \mathbb{R}^p$ and seek to recover the evolution rule from $\{x(t_k)\}$.

Dynamical System Identification



- We parameterize $v(x;\theta)$ by $\theta \in \mathbb{R}^p$ and seek to recover the evolution rule from $\{x(t_k)\}$.
- Common approaches typically operate directly on trajectories, e.g., Neural ODE (Chen et al., 2018) and SINDy (Brunton et al., 2016).

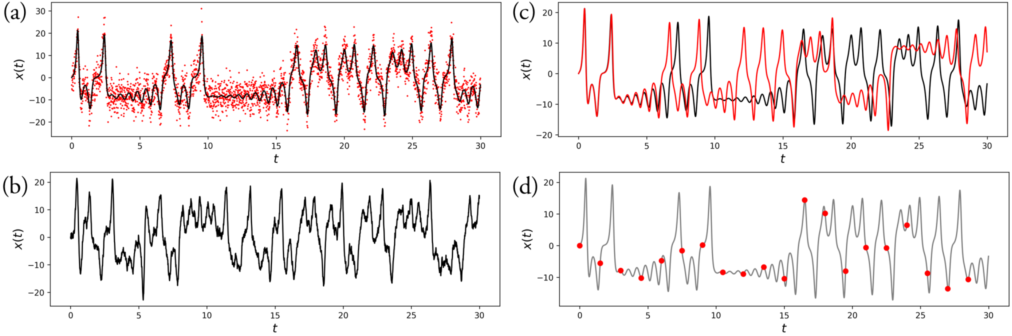
Challenges in System Identification

(a) **Extrinsic noise:** $\{x(t_k) + \varepsilon(t_k)\}_{k=0}^{N-1}$

(b) **Intrinsic noise:** $dX_t = v(X_t)dt + \sigma(X_t)dW_t$

(c) **Chaos:** $\|\Delta x(t_k)\| \approx e^{\lambda t_k} \|\Delta x(t_0)\|$

(d) **Slow sampling:** $t_k - t_{k-1} \gg \lambda^{-1}$



Invariant Measures for System Identification

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- Convert $\{x(t_k)\}_{k=0}^{N-1}$ into the **occupation measure** $\rho^* \in \mathcal{P}(X)$, where

$$\rho^*(B) := \frac{1}{N}\sum_{k=0}^{N-1} \chi_B(x(t_k)), \qquad \forall B \in \mathscr{B}(X), \qquad \chi_B(x) := \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$

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- Reformulate system identification as optimization over a probability space:

$$\min_{\theta \in \Theta} \mathcal{J}(\theta), \qquad \mathcal{J}(\theta) = \mathcal{D}(\rho(\theta), \rho^*), \qquad \mathcal{D} : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty).$$

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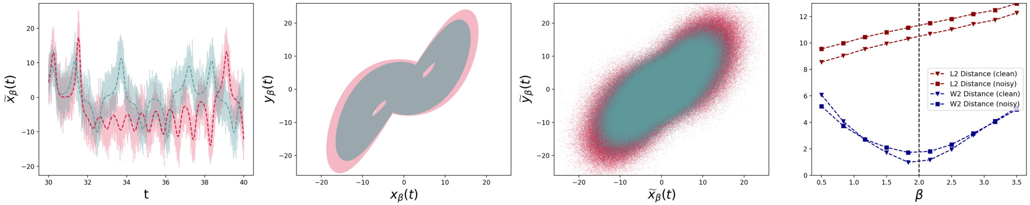

Reformulation as PDE-Constrained Optimization

- For efficient gradient computation, approximate $\theta \mapsto \rho(\theta)$ as $M(\theta)\rho(\theta) = \rho(\theta)$, where $M(\theta)$ is a Markov matrix arising from a finite-volume discretization of the Fokker–Planck equation.

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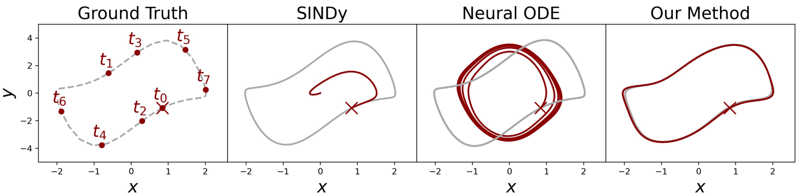

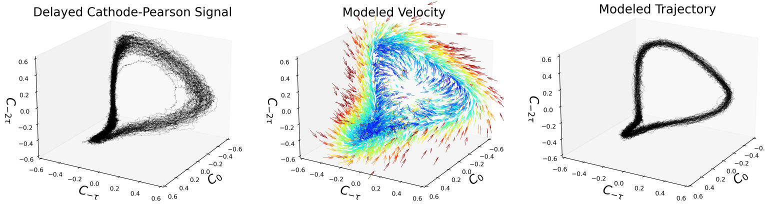

J., Martin, R., Yang, Y. (2023). Learning dynamics on invariant measures using PDE-constrained optimization. Chaos, 33(6).

Related Work

1. *Invariant measures for system identification:*
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   - Muskulus and Verduyn-Lunel, 2011
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   - Park et al., 2024

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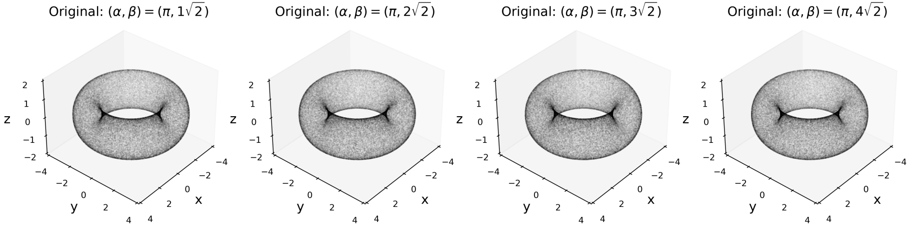

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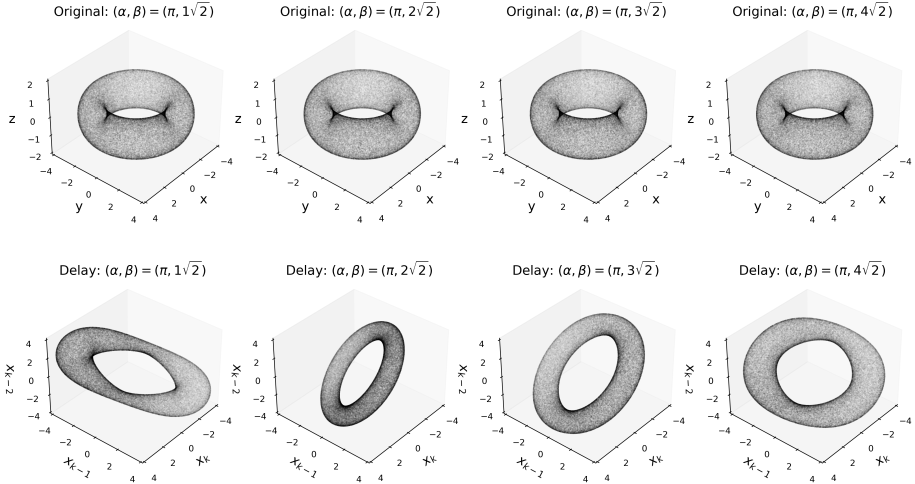

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- We choose the coordinate transformation as **Takens' time-delay map**

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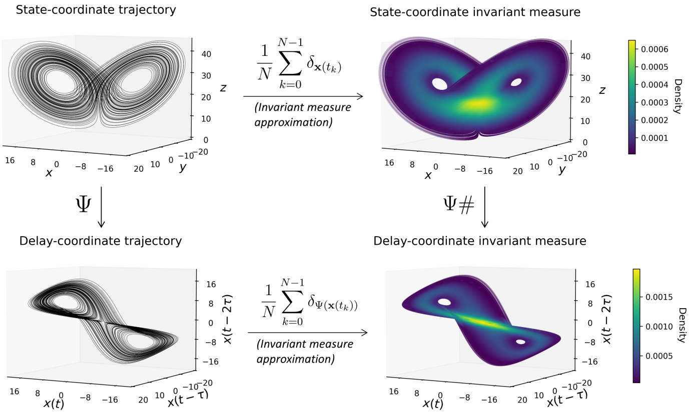
- Is $\Psi^{(m)}_{(y,T)}$ invertible? Yes, under mild conditions given by Takens' embedding theorem.

Invariant Measures in Time-Delay Coordinates (cont.)

Definition

If μ is T-invariant, then $\hat{\mu}^{(m)}_{(y,T)} := \Psi^{(m)}_{(y,T)} \# \mu$ is the *delay-coordinate invariant measure.*



Unique Identifiability: Main Theoretical Results

Invariant measure

$$T\#\mu \;=\; \mu$$

No uniqueness

One delay-coordinate invariant measure

$$\hat{\mu}^{(m+1)}_{(y,T)} \;=\; \Psi^{(m+1)}_{(y,T)}\#\mu$$

Conjugacy on supp(μ)

***m* delay-coordinate invariant measures**

$$\left\{\hat{\mu}^{(m+1)}_{(y_i,T)}\right\}_{i=1}^{m}$$

Uniqueness on supp(μ)

Unique Identifiability: Main Theoretical Results (cont.)

Theorem

The equality $\hat{\mu}^{(m+1)}_{(y,T)} = \hat{\nu}^{(m+1)}_{(y,S)}$ implies $T|_{\mathrm{supp}(\mu)}$ and $S|_{\mathrm{supp}(\nu)}$ are topologically conjugate, for almost every $y \in C^1(U, \mathbb{R})$.

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Theorem

The conditions

1. *there exists* $x^* \in B_{\mu,T} \cap \mathrm{supp}(\mu)$, *such that* $T^k(x^*) = S^k(x^*)$ *for* $1 \leq k \leq m-1$, *and*
2. $\hat{\mu}^{(m+1)}_{(y_j,T)} = \hat{\mu}^{(m+1)}_{(y_j,S)}$ *for* $1 \leq j \leq m$, *where* $Y := (y_1, \ldots, y_m)$ *is a vector-valued observable,*

imply that $T = S$ *everywhere on* $\mathrm{supp}(\mu)$, *for almost every* $Y \in C^1(U, \mathbb{R}^m)$.

Numerical Example: Lorenz-63 System

We attempt to learn the dynamics of the Lorenz-63 system using the loss functions

$$\mathcal{J}_1(\theta) := \mathcal{D}(T_\theta \# \mu^*, T^* \# \mu^*), \quad \mathcal{J}_2(\theta) := \mathcal{D}(T_\theta \# \mu^*, T^* \# \mu^*) + \mathcal{D}(\Psi_\theta \# \mu^*, \Psi^* \# \mu^*).$$

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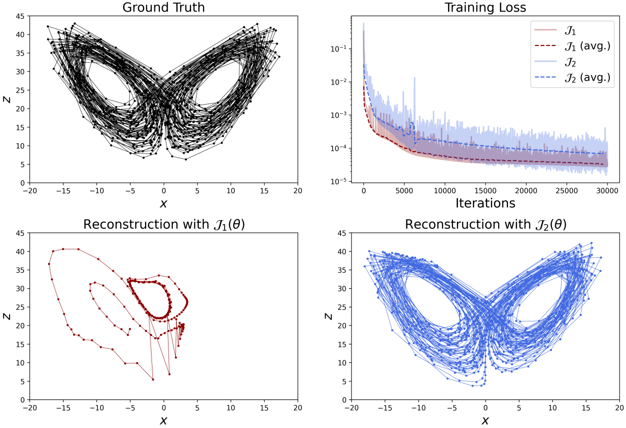

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