Dynamical System Identification via Invariant Measures in Time-Delay Coordinates

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Abstract

Invariant measures are widely used to **compare chaotic dynamical systems**, as they offer **robustness** to noisy data, uncertain initial conditions, and irregular sampling. However, systems with **distinct transient dynamics** may still exhibit the **same asymptotic statistical behavior**. To address this limitation, we propose comparing invariant measures in **time-delay coordinate systems**. In particular, we prove that a single invariant measure in time-delay coordinates can **identify** the underlying dynamics, up to a **topological conjugacy**. Our findings significantly improve the utility of invariant measures for system identification and broaden the scope of **measure-theoretic approaches** for modeling dynamical systems.

Introduction

- Problem statement: *Consider a dynamical system $T_\theta : X \to X$, with unknown parameters $\theta \in \Theta \subseteq \mathbb{R}^p$. Given trajectory data $\{T^k_{\theta^*}(x)\}_{k=1}^N$, recover $\theta_* \in \Theta$.*

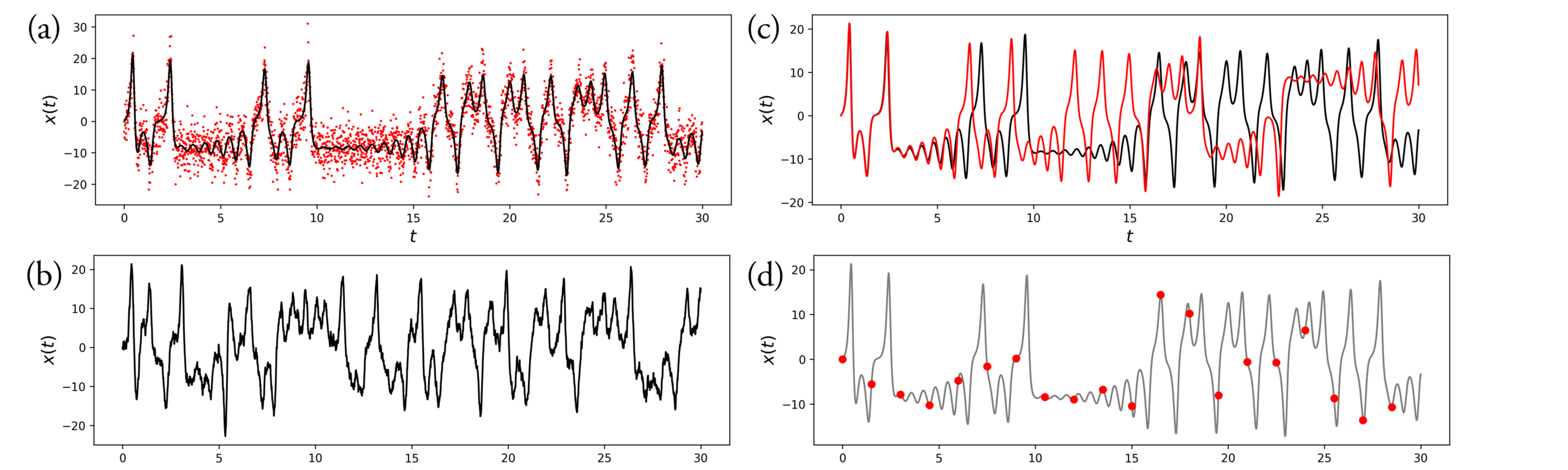

Figure 1. Illustration of the challenges one may encounter during the parameter identification problem: (a) extrinsic noise, (b) intrinsic noise, (c) chaos, and (d) slow sampling.

Motivation

- Approach Using Invariant Measures: Several recent works treat the *invariant measure* as inference data:

$$\rho(\theta^*) \approx \frac{1}{N}\sum_{k=0}^{N-1} \delta_{T^k_{\theta^*}(x)} \in \mathcal{P}(X),$$

and solve the inverse problem via the optimization

$$\min_{\theta\in\Theta} \mathcal{J}(\theta), \qquad \mathcal{J}(\theta) = \mathcal{D}(\rho(\theta), \rho(\theta^*)),$$

where $\mathcal{D} : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty)$ is a metric or divergence on $\mathcal{P}(X)$, such as the **Wasserstein distance**, and $\theta \mapsto \rho(\theta)$ is the parameter to invariant measure map [1, 2, 3].

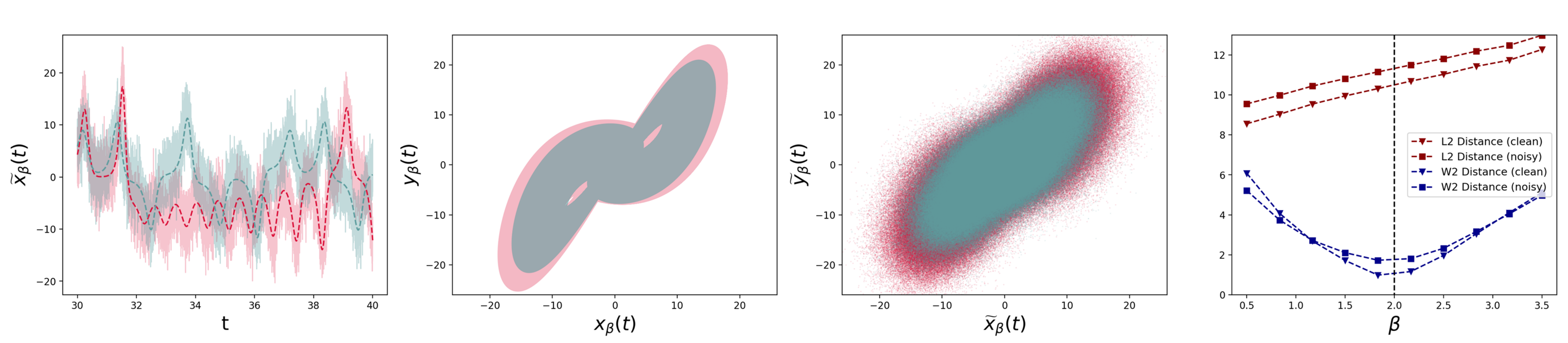

Figure 2. Using the invariant measure to compare nearby dynamical systems results in a meaningful optimization landscape, whereas a trajectory-based loss does not.

- Fundamental Difficulty (Non-uniqueness): Infinitely many systems can share the same invariant measure, i.e., $\theta \mapsto \rho(\theta)$ need not be injective.
- Motivation: Introduce **time-dependence** into the invariant measure.

Our Contributions

- We rigorously show that invariant measures in time-delay coordinates offer substantial benefits over state-coordinate invariant measures.

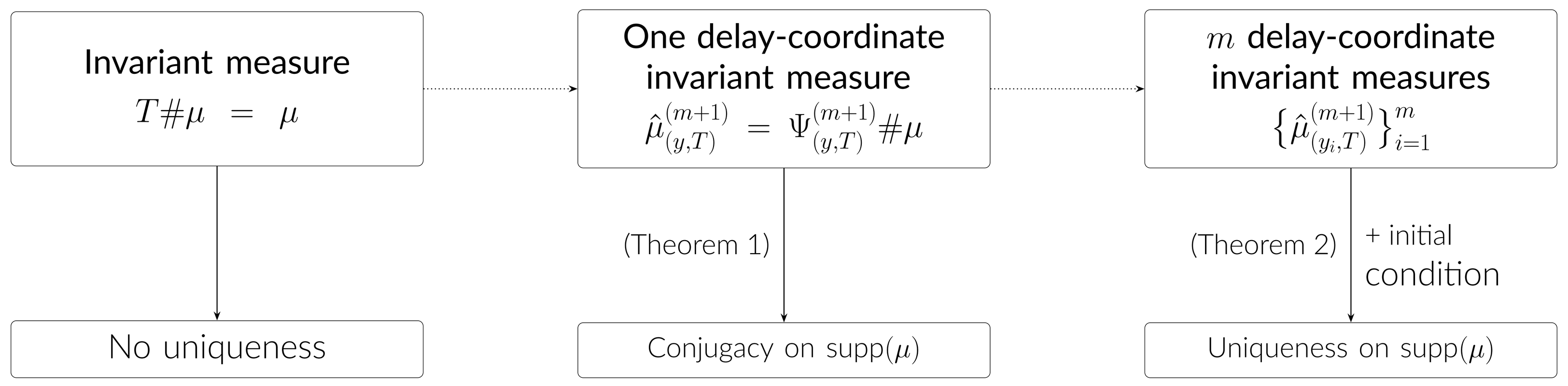

Figure 3. Flowchart of our main results.

- Figure 4 shows that invariant measures in time-delay coordinates have greater sensitivity to the dynamics than state-coordinate invariant measures.

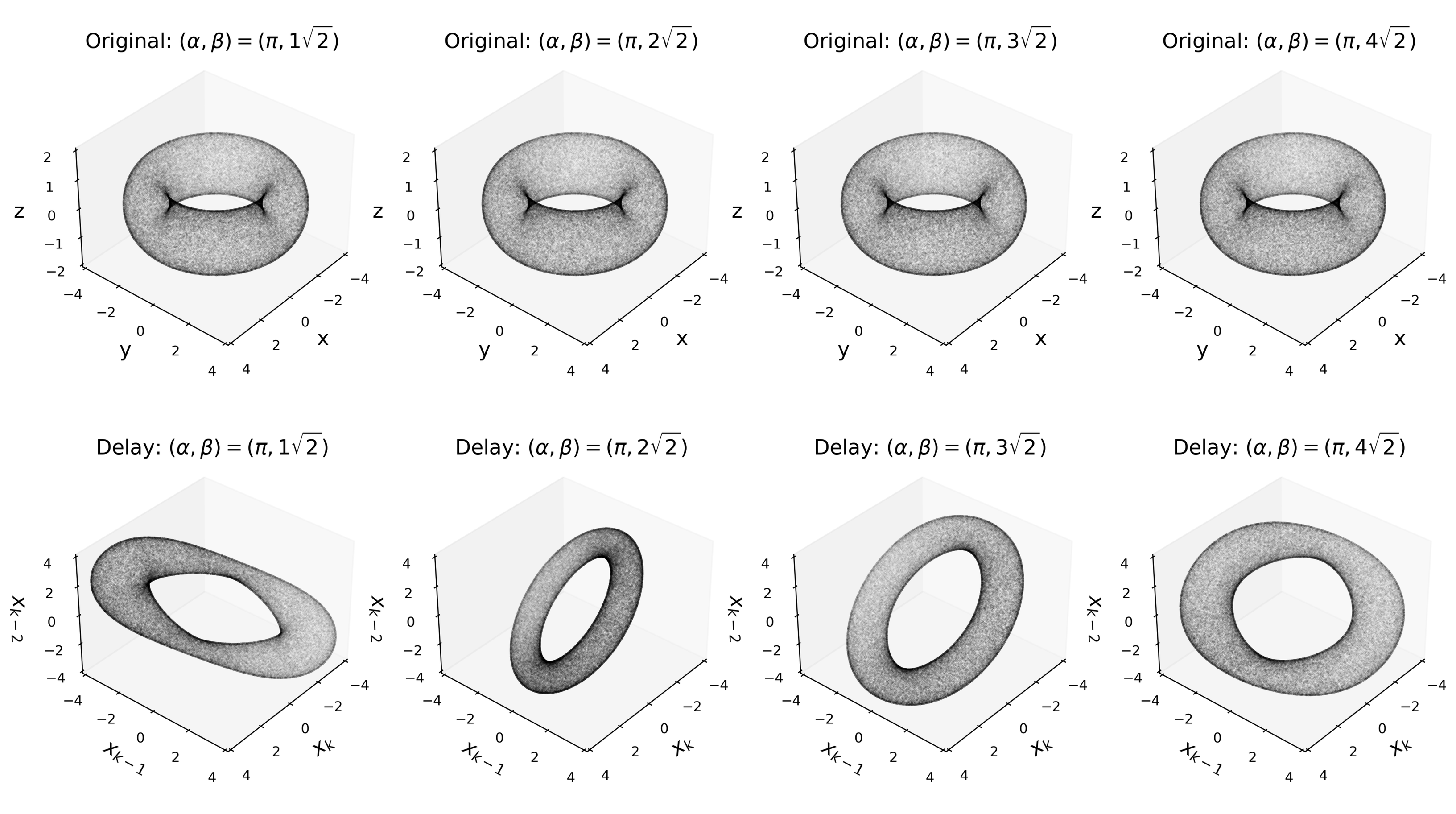

Figure 4. Illustration of the difference between state-coordinate invariant measures and delay coordinate invariant measures for the torus rotation $T_{\alpha,\beta}(z_1, z_2) = (z_1 + \alpha, z_2 + \beta) \pmod 1$.

Invariant Measures Under Conjugacy

- Invariant Measure: $\mu \in \mathcal{P}(X)$ is T-invariant if $T\#\mu = \mu$, i.e., $\mu(T^{-1}(B)) = \mu(B)$, for all Borel sets $B \subseteq X$.
- Topological Conjugacy: $T \in C(X, X)$ and $S \in C(Y, Y)$ are topologically conjugate if $S = h \circ T \circ h^{-1}$ for some homeomorphism $h : X \to Y$.
- Proposition 1: *If μ is T-invariant and $S = h \circ T \circ h^{-1}$, then $h\#\mu$ is S-invariant.*
- Key Insight: If $T, S : X \to X$ and μ is T-invariant and S-invariant, we instead study $h_T\#\mu \in \mathcal{P}(\mathbb{R}^m)$ and $h_S\#\mu \in \mathcal{P}(\mathbb{R}^m)$, where $h_T, h_S : X \to \mathbb{R}^m$ are **map-dependent coordinate transformations.**
- Time-delay map: We show that using the delay map

$$\Psi^{(m)}_{(y,T)}(x) = (y(x), y(T(x)), \ldots, y(T^{m-1}(x))) \in \mathbb{R}^m, \qquad y : X \to \mathbb{R} \tag{1}$$

from Takens' embedding theory is a strong choice of coordinate transformation. Note that (1) is injective on a compact subset $A \subseteq \mathbb{R}^n$ when m is twice the intrinsic dimension of A; see [4].

Delay-Coordinate Invariant Measures

- Definition: If μ is T-invariant, then $\hat{\mu}^{(m)}_{(y,T)} := \Psi^{(m)}_{(y,T)}\#\mu \in \mathcal{P}(\mathbb{R}^m)$ is the corresponding *delay-coordinate invariant measure.*
- Connection to OT: When $\Psi^{(m)}_{(y,T)}$ is an embedding, then $\Psi^{(m)}_{(y,T)}\#$ is an embedding of the corresponding Wasserstein space [5].

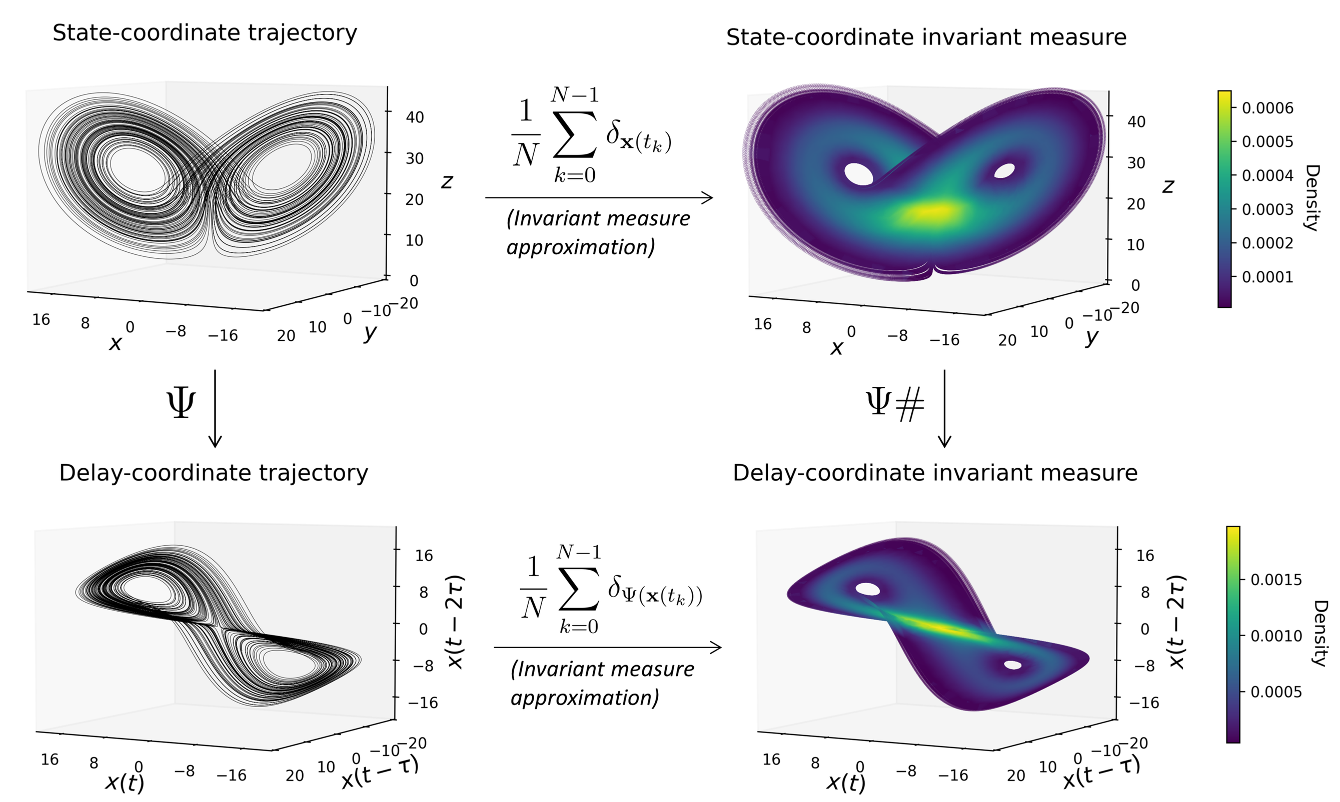

Figure 5. Visualization of the state-coordinate and delay-coordinate invariant measures for the Lorenz-63 system. Both can be easily approximated from trajectory data.

Main Theorems (simplified)

- Theorem 1 (One measure $\implies$ conjugacy): *If $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $\nu \in \mathcal{P}(\mathbb{R}^n)$ are T-invariant and S-invariant, respectively, then $\hat{\mu}^{(m+1)}_{(y,T)} = \hat{\nu}^{(m+1)}_{(y,S)}$* ***implies that the maps $T|_{\mathrm{supp}(\mu)}$ and $S|_{\mathrm{supp}(\nu)}$ are topologically conjugate.***
- Theorem 2 (More measures $\implies$ uniqueness): *If $\mu \in \mathcal{P}(\mathbb{R}^n)$ is T-invariant and S-invariant, then equality $\hat{\mu}^{(m+1)}_{(y_i,T)} = \hat{\mu}^{(m+1)}_{(y_i,S)}$ for $1 \le i \le m$,* ***implies that the maps $T|_{\mathrm{supp}(\mu)}$ and $S|_{\mathrm{supp}(\mu)}$ are equal****, provided a suitable initial condition holds.*

Conclusion

While many researchers have leveraged the use of the **state-coordinate invariant measure** to perform system identification, this approach **cannot distinguish** between large classes of models which all share the **same asymptotic behavior**. To this end, we have developed a theoretical framework that justifies the use of **invariant measures in time-delay coordinates** for comparing dynamical systems.

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Our manuscript is available upon request [6]

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