Unique Recovery of Transport Maps and Vector Fields from Finite Measure-Valued Data

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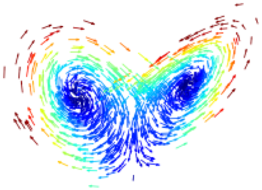

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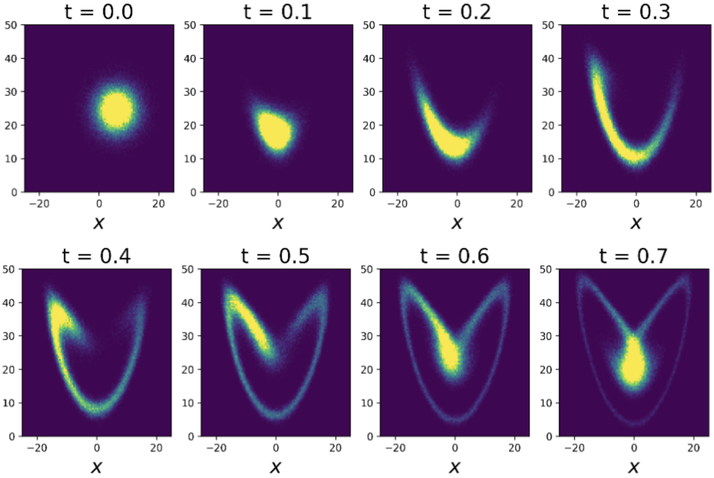

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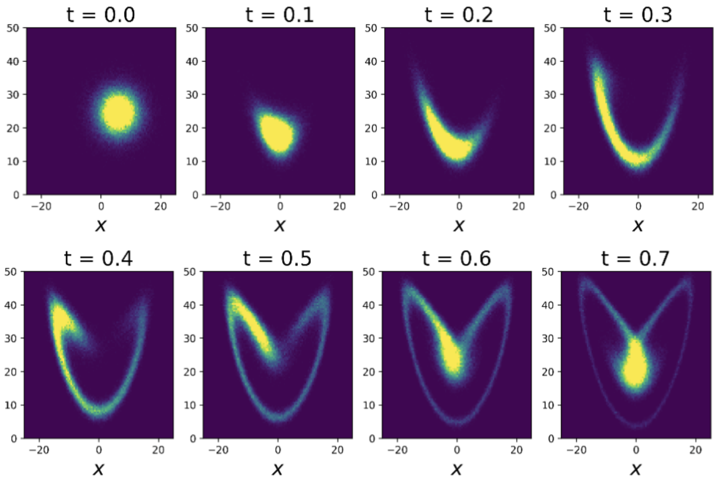

Can we **uniquely** recover the dynamics from m density snapshots?

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Pushforward uniqueness

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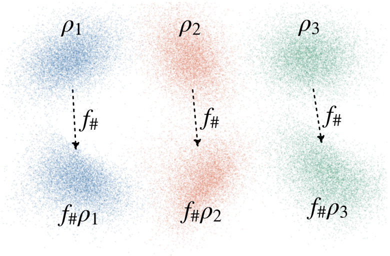

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Infinitesimal case $\longrightarrow$

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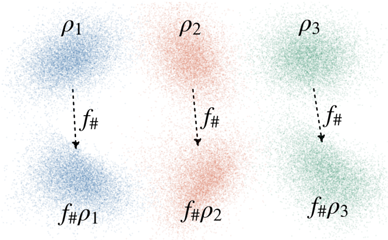

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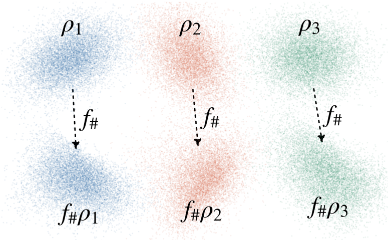

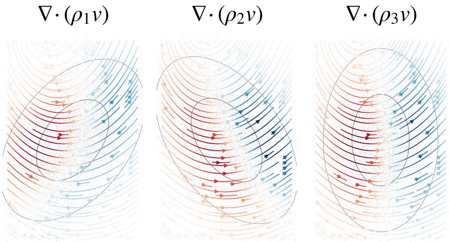

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$\Downarrow$ $m > 2d+1$

Thm: Yes, for a generic set of densities $\boldsymbol{D}$.

$\xrightarrow{\text{Infinitesimal case}}$

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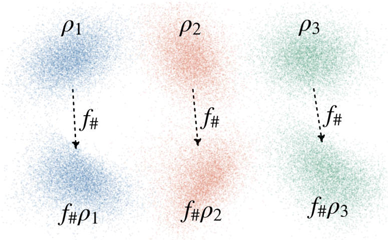

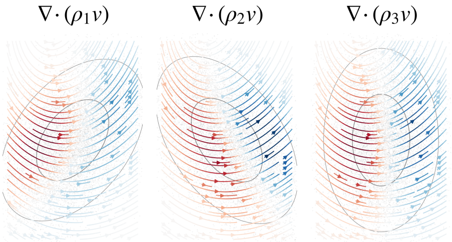

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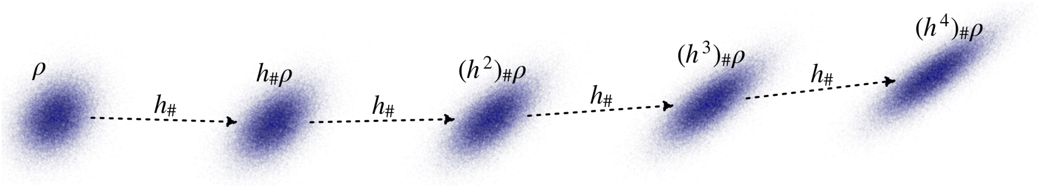

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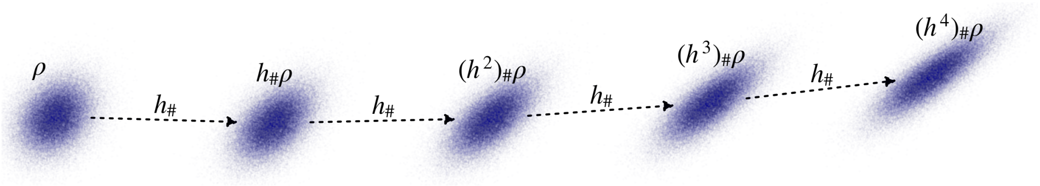

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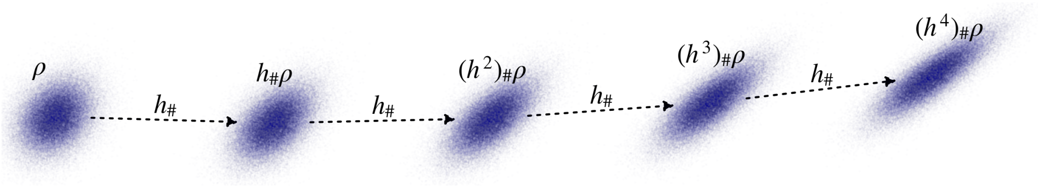
Applications

- A new metric comparing pushforward actions
- Data-driven dynamics
- PDE inverse problems
- Generative models
- Measure-valued data fitting



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If $\Phi = (\rho_1/\rho_m, \ldots, \rho_{m-1}/\rho_m)$ is an embedding **(!)**, then $\Phi \circ f^{-1} = \Phi \circ g^{-1}$ and $f = g$. $\square$

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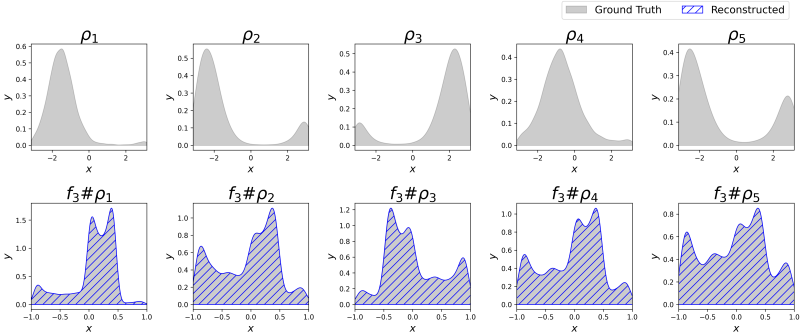

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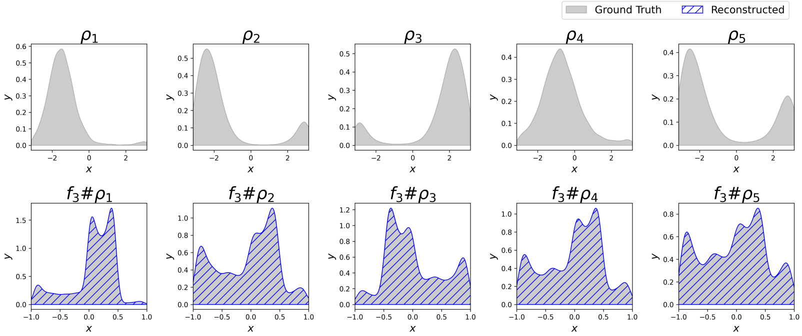

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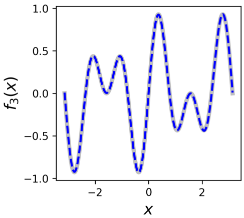

Figure: Reconstructed function

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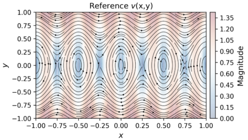

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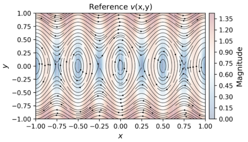

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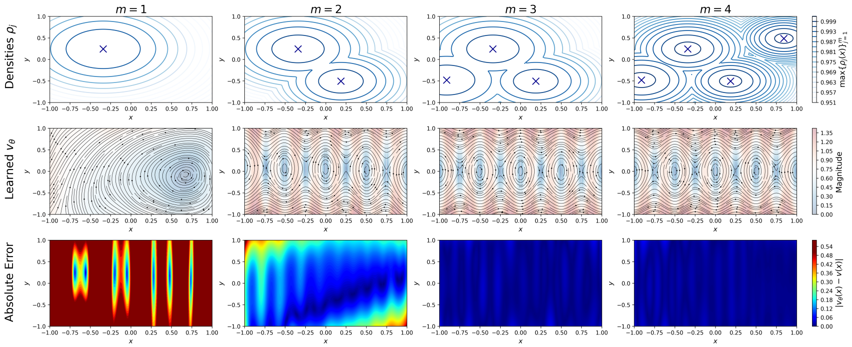

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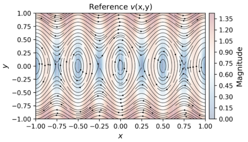

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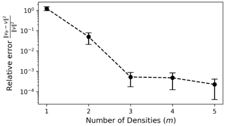
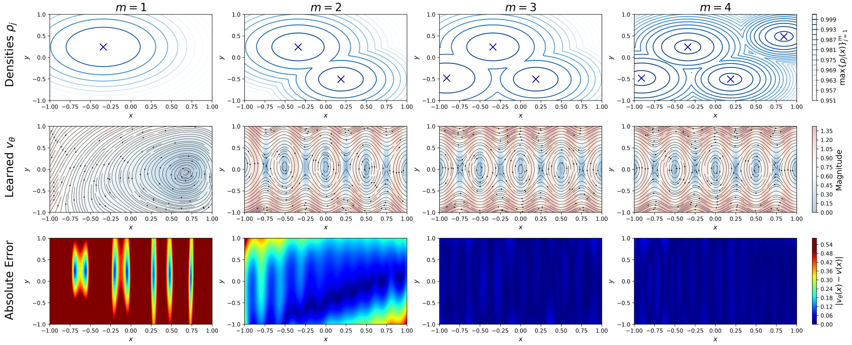


Figure: Error as a function of m.



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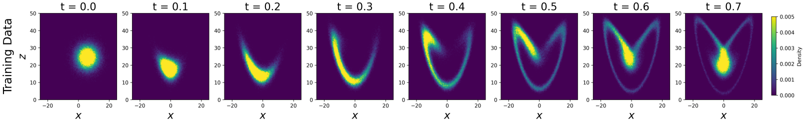

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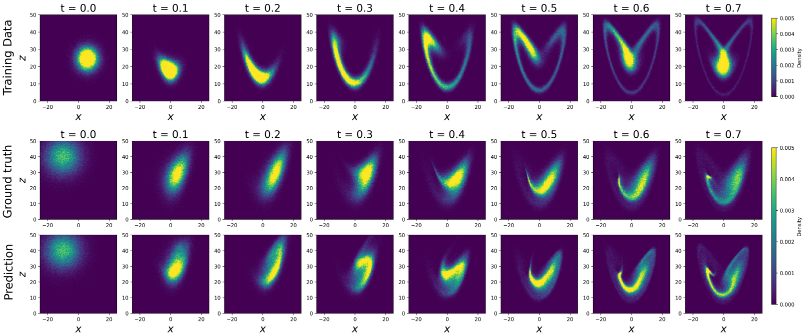
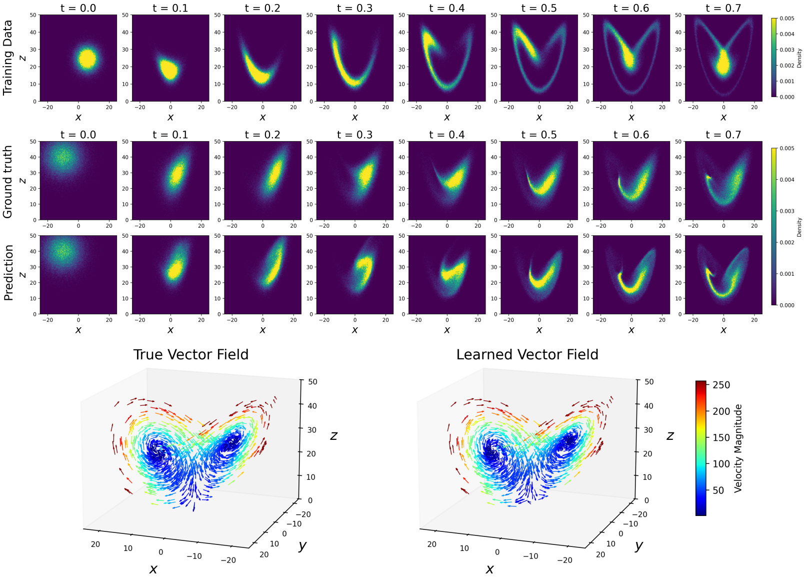


Lorenz-63 Pushforward Recovery



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Thank you for your attention!

Botvinick-Greenhouse, J., & Yang, Y. (2026).
On the Unique Recovery of Transport Maps and Vector Fields from Finite Measure-Valued Data.
arXiv preprint arXiv:2604.07671.

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